Fault-tolerant logical gates in quantum error-correcting codes

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arXiv:1408.1720

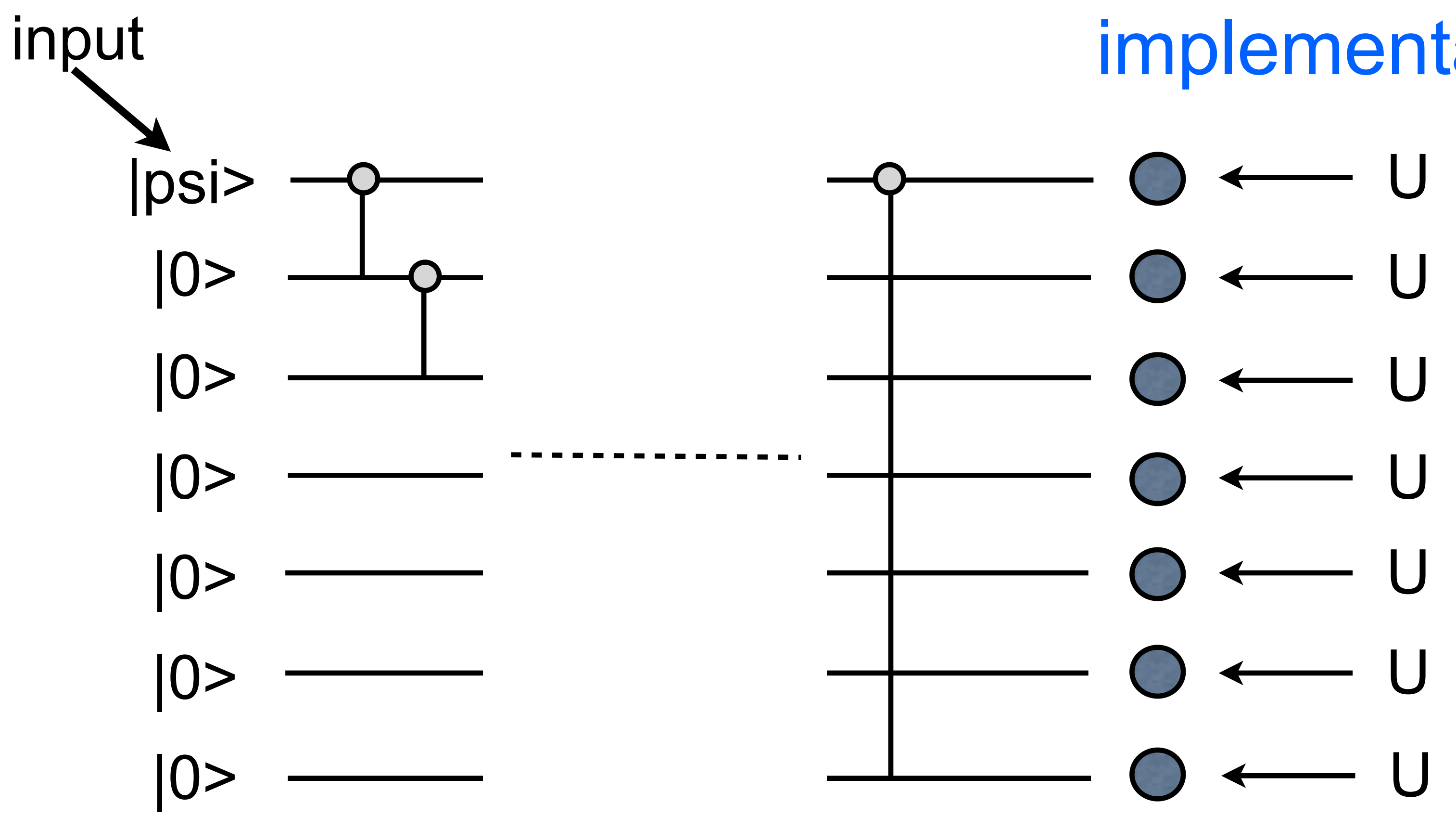
Phys Rev A xxxxxxx

Jan 2015 @ QIP (Sydney, Australia)

Fault-tolerant logical gates

- How do we implement a logical gate fault-tolerantly ?

Ideally, by transversal implementation



encoding circuit

The Eastin-Knill theorem (2008)

- Transversal logical gates are not universal for QC

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Restrictions on Transversal Encoded Quantum Gate Sets

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(Received 28 November 2008; published 18 March 2009)

Transversal gates play an important role in the theory of fault-tolerant quantum computation due to their simplicity and robustness to noise. By definition, transversal operators do not couple physical subsystems within the same code block. Consequently, such operators do not spread errors within code blocks and are, therefore, fault tolerant. Nonetheless, other methods of ensuring fault tolerance are required, as it is invariably the case that some encoded gates cannot be implemented transversally. This observation has led to a long-standing conjecture that transversal encoded gate sets cannot be universal. Here we show that the ability of a quantum code to detect an arbitrary error on any single physical subsystem is incompatible with the existence of a universal, transversal encoded gate set for the code.

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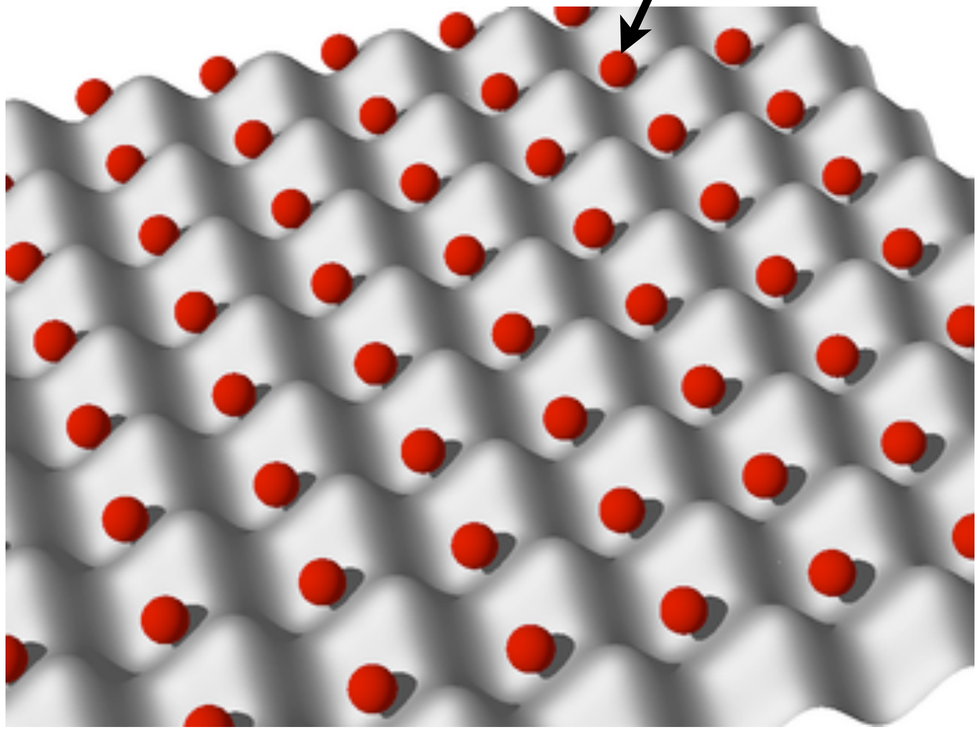
Don’t panic ! Fault-tolerant computation is still possible.

The Bravyi-Koenig theorem (2012)

- Under a more physically realistic setting

Logical gate U : low-depth unitary gate (i.e. Local unitary)



D-dim lattice

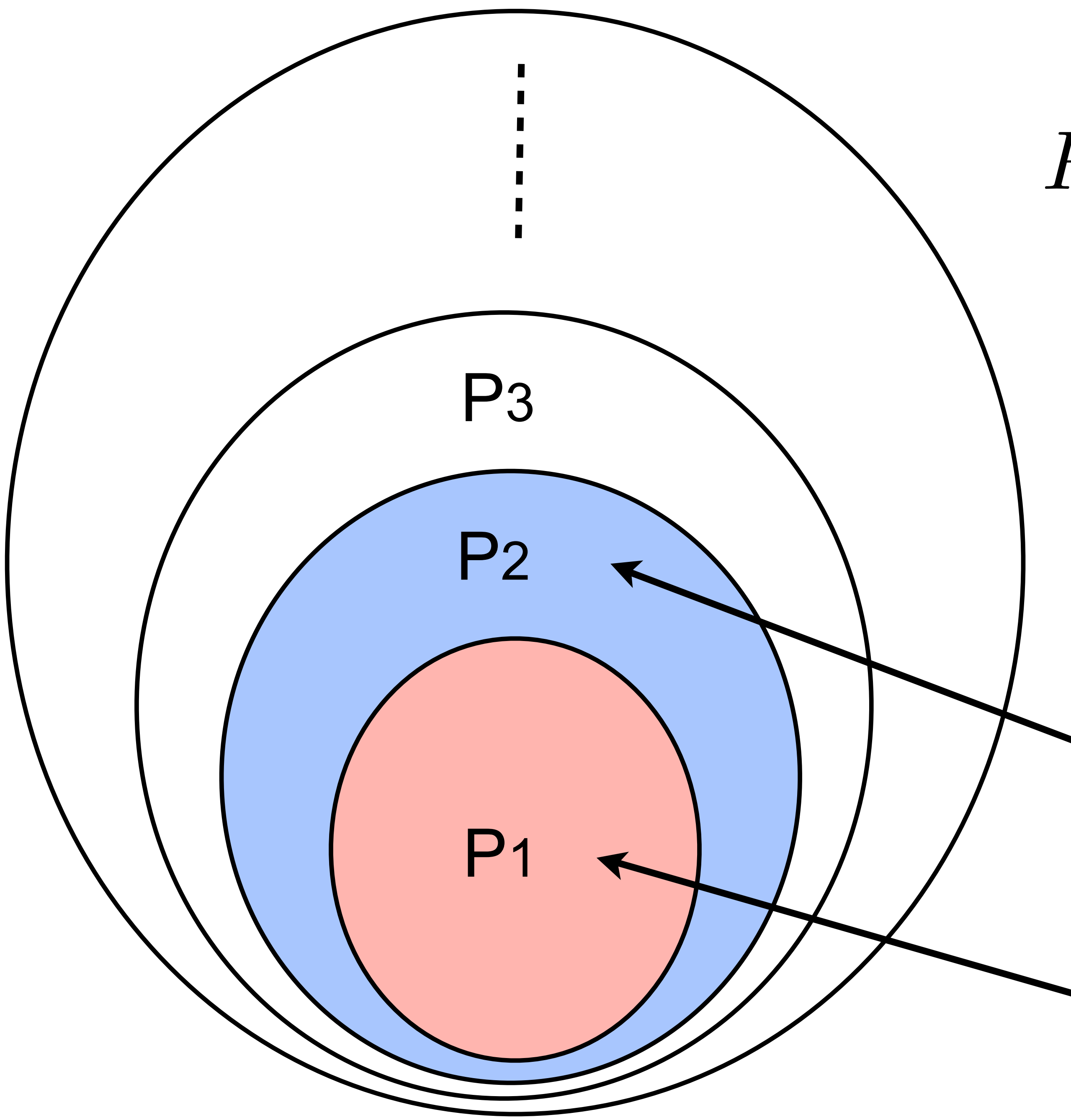
Theorem

- For a stabilizer Hamiltonian in D dim, fault-tolerantly implementable gates are restricted to the D-th level of the Clifford hierarchy.

???

Clifford hierarchy (Gottesman & Chuang)

Sets of unitary transformations on N qubits



$$P_m \ \text{Pauli} \ P_m^\dagger = P_{m-1}$$

$$P_3 \ \text{Pauli} \ P_3^\dagger = P_2$$

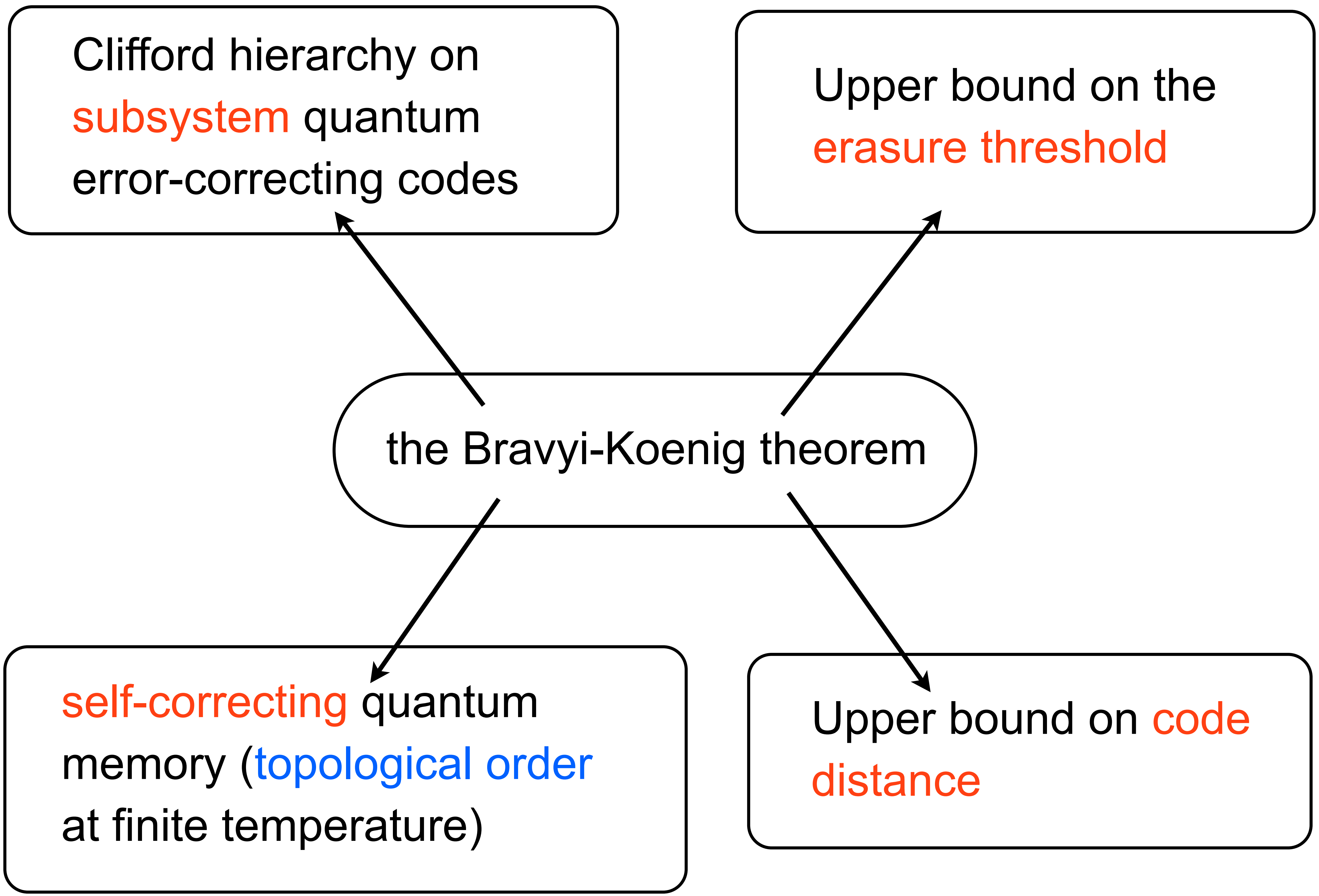
$$P_2 \ \text{Pauli} \ P_2^\dagger = P_1$$

Clifford gates
CNOT, Hadamard, R_2

Pauli

Pauli operators
X,Y,Z

Plan of the talk



Plan of the talk

Clifford hierarchy on subsystem error-correcting codes

Upper bound on the erasure threshold

the Bravyi-Koenig theorem

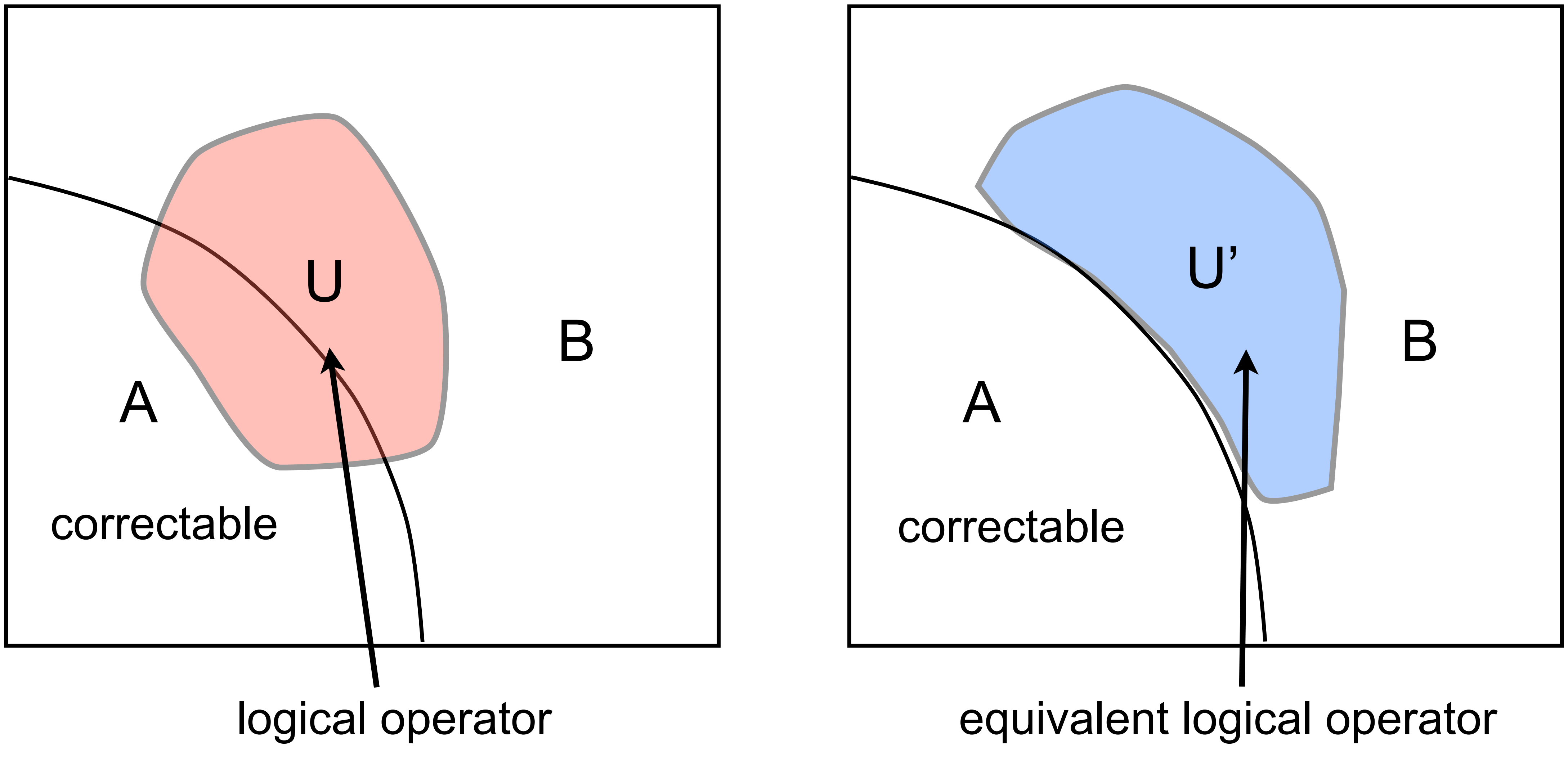
self-correcting memory (at finite temperature)

Upper bound on distance

Logical operator cleaning

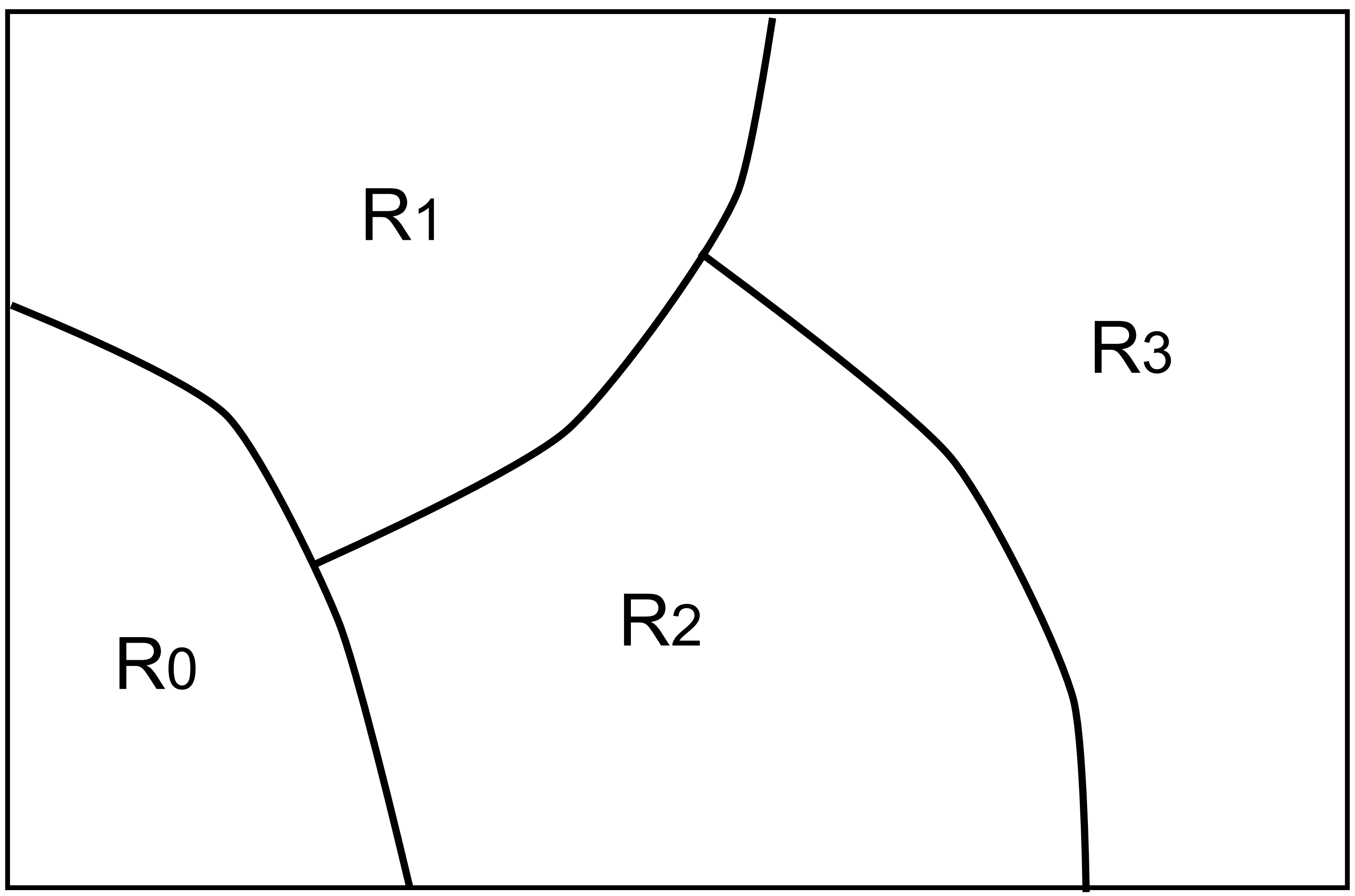
- A logical operator can be “cleaned” from a correctable region.

A “correctable region” supports no logical operator.



Lemma [Hierarchy]

Consider a partition of the entire system into m+1 regions, denoted by $R_0, R_1, \ldots, R_m$. If all R_j's are correctable, then transversal logical gates are restricted to m-th level P_m of the Clifford hierarchy.



(eg)

4 correctable regions

↓

P_3

- Consider arbitrary Pauli logical operators $V_0, V_1, \ldots V_m$.

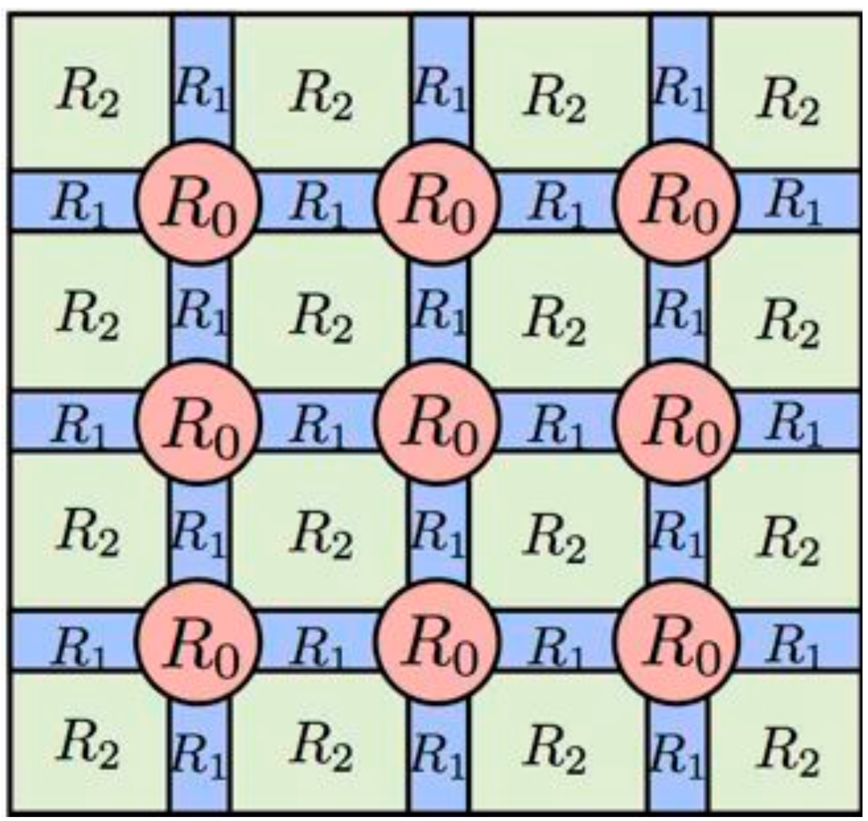
	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_0		✓	✓	⋯	✓	✓	
V_1	✓		✓	⋯	✓	✓	Pauli
⋮				⋮			
V_m	✓	✓	✓	⋯	✓		
U_0	✓	✓	✓	⋯	✓	✓	P_m (goal)
$U_1=K(U_0,V_0)$		✓	✓	⋯	✓	✓	P_{m-1}
$U_2=K(U_1,V_1)$			✓	⋯	✓	✓	P_{m-2}
⋮				⋮			
$U_{m-1}=K(U_{m-2},V_{m-2})$				⋯	✓	✓	P_1 (Pauli)
$U_m=K(U_{m-1},V_{m-1})$				⋯		✓	Complex phase

commutator : $K(A,B)=ABA^*B^*$

Proof of the Bravyi-Koenig theorem

- We can split D-dimensional system into D+1 correctable regions.

(eg) 2 dim

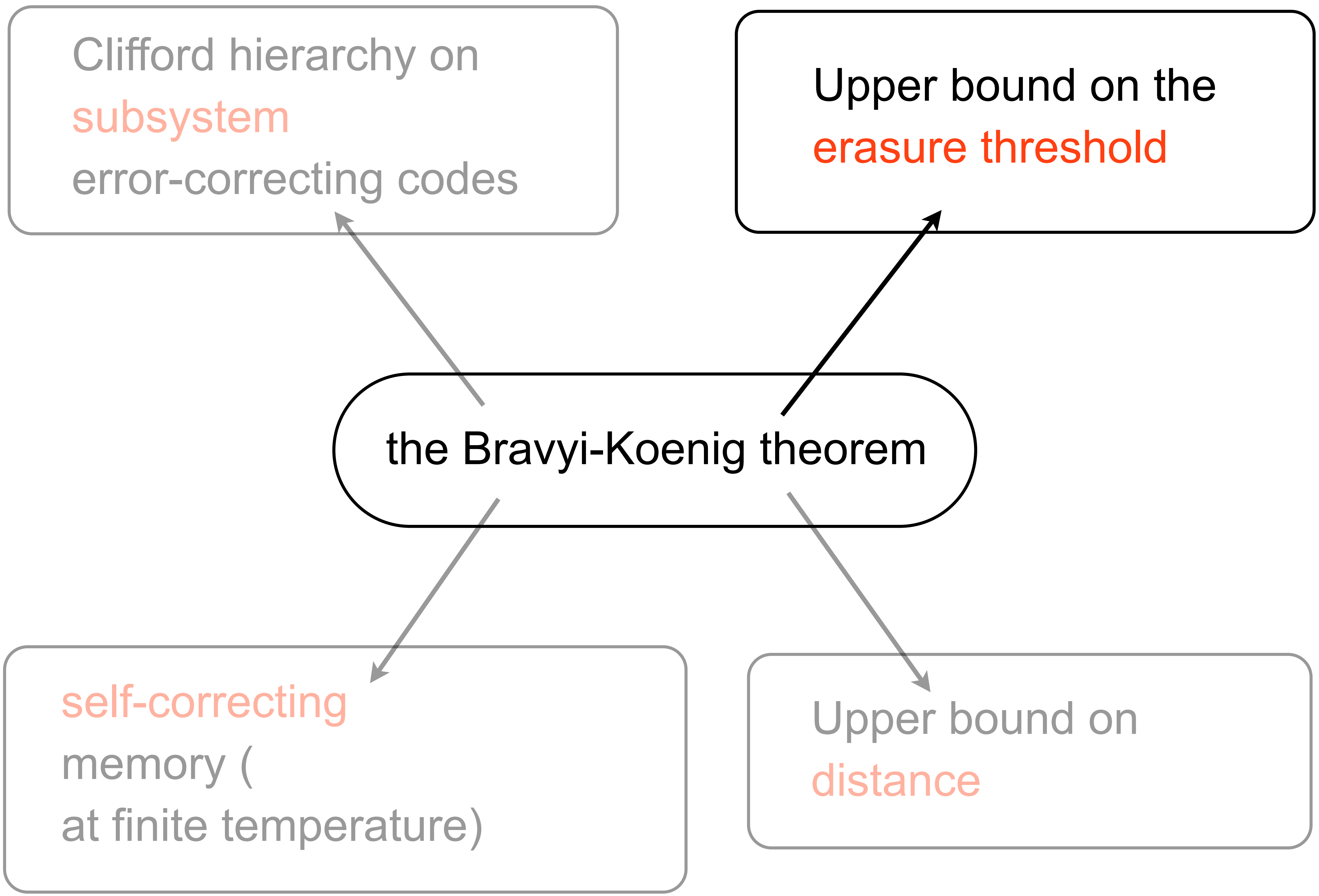


Fault-tolerant gates are in P_2

*Union of spatially disjoint correctable regions = correctable region

*This is not the case for subsystem codes.

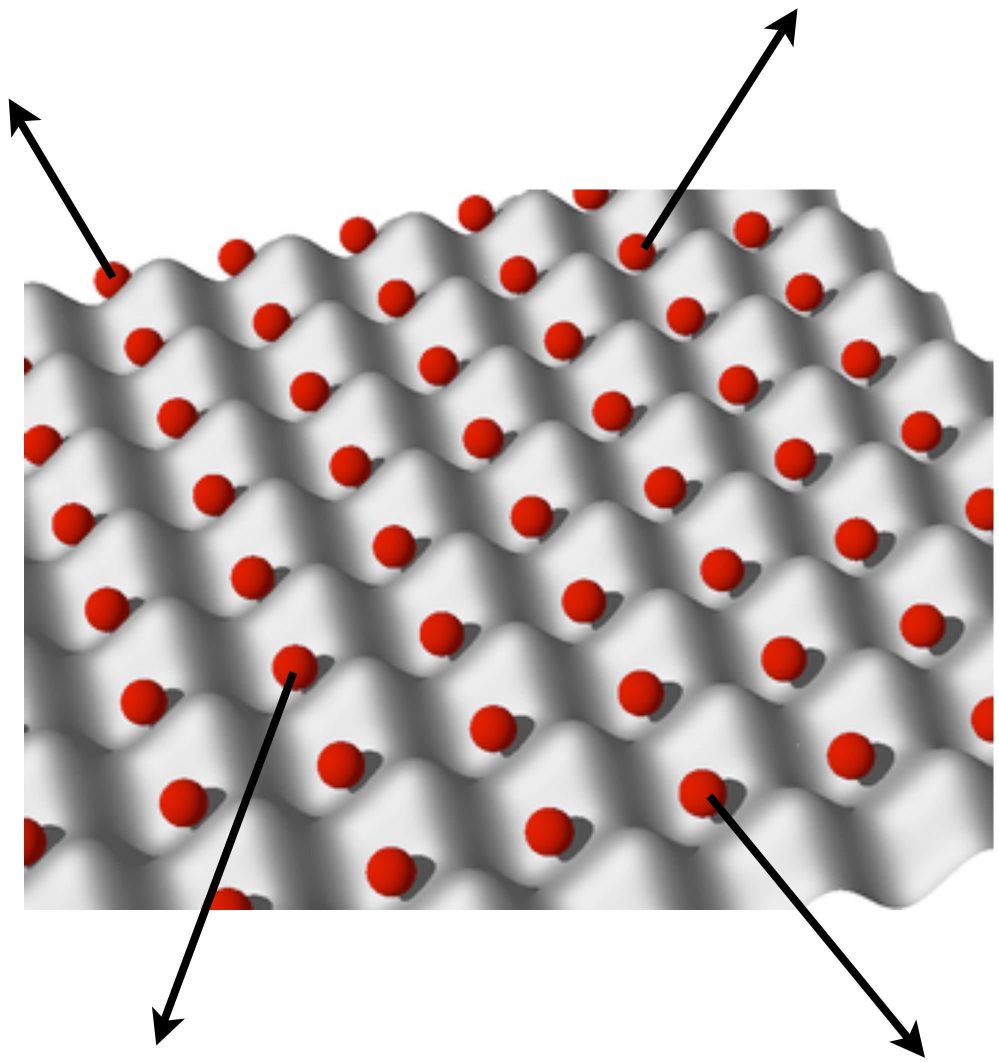
Plan of the talk



Erasure Threshold

- Some qubits may be lost (removal errors)...

eg) escape from the trap



$$p < p_{\text{loss}} \quad \Rightarrow \quad \text{Logical qubit is safe}$$

erasure threshold

$$p_{\text{error}} < p_{\text{loss}}$$

against depolarizing error

Theorem. [Loss threshold] *Suppose we have a family of subsystem codes with a loss tolerance* $p_l > 1/n$ *for some natural number* n. *Then, any transversally implementable logical gate must belong to* $\mathcal{P}_{n-1}$.

$$\mathcal{P}_n \text{ logical gate} \quad \Rightarrow \quad p_\ell \leq 1/n.$$

Proof sketch

- Assign each qubit to n regions uniformly at random

  $R_1, R_2, \ldots R_n$

- All the regions are cleanable since $p_l > 1/n$
- Transversal gates must be in P_{n-1}

Theorem. [Loss threshold] *Suppose we have a family of subsystem codes with a loss tolerance $p_l > 1/n$ for some natural number n. Then, any transversally implementable logical gate must belong to $\mathcal{P}_{n-1}$.*

$$\mathcal{P}_n \text{ logical gate} \quad \Rightarrow \quad p_\ell \geq 1/n.$$

Remarks

- Toric code has p=1/2 threshold (related to percolation).

  It has a transversal P_2 gate (CNOT gate)

- A family of codes with growing n is not fault-tolerant.

- Topological color code in D-dim has P_D gate, so its loss threshold is less than 1/D.

Theorem. **[Loss threshold]** *Suppose we have a family of subsystem codes with a loss tolerance* $p_l > 1/n$ *for some natural number* n. *Then, any transversally implementable logical gate must belong to* $\mathcal{P}_{n-1}$.

$$\mathcal{P}_n \text{ logical gate} \quad \Rightarrow \quad p_\ell \geq 1/n.$$

<u>One additional remark</u> (due to Leonid Pryadko)

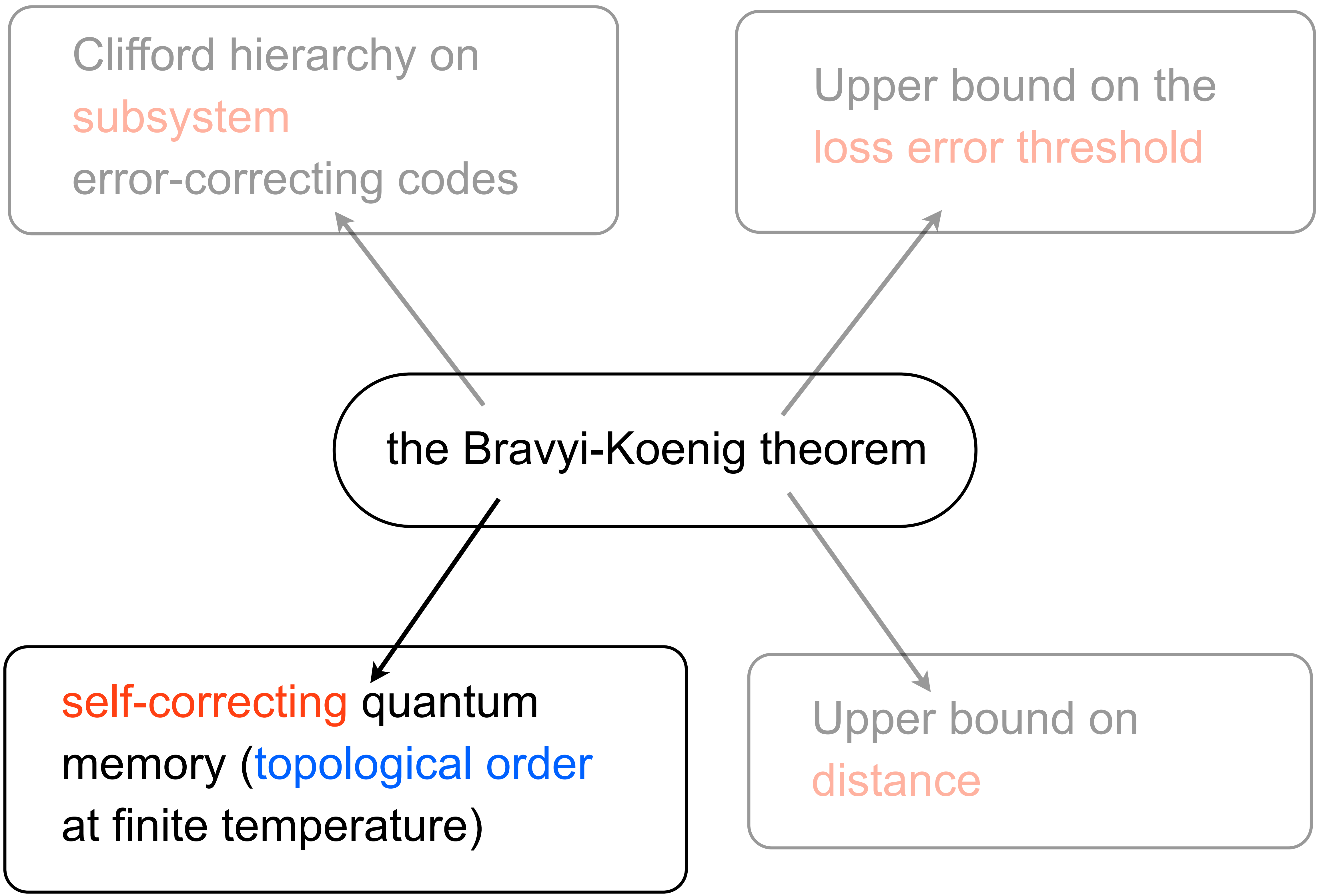
Consider a stabilizer code with at most k-body generators.

If the code has transversal P_D logical gate, then

k > O(D)

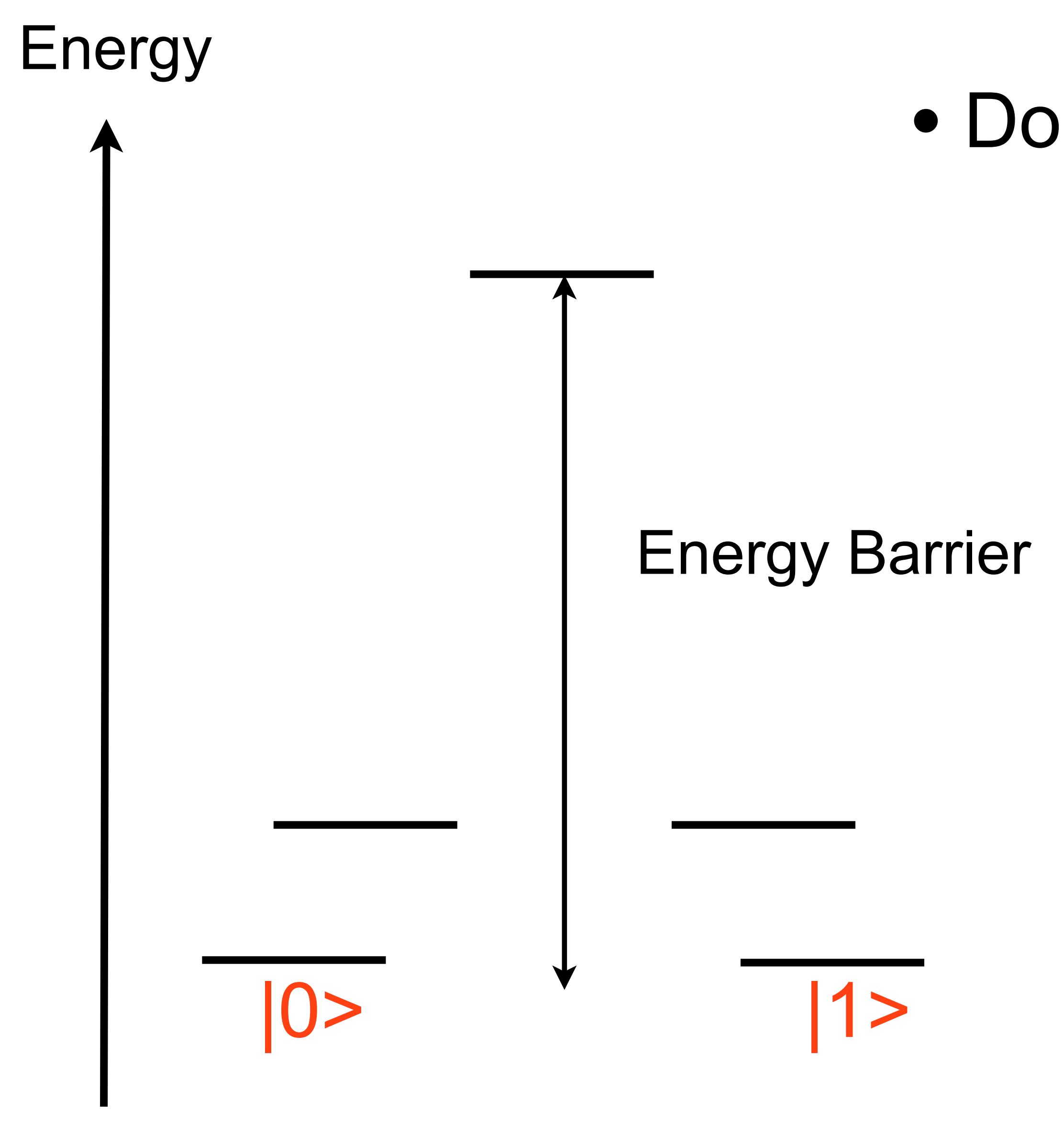
- D-dim color code is ~2^D body. Fewer-body code?

Plan of the talk

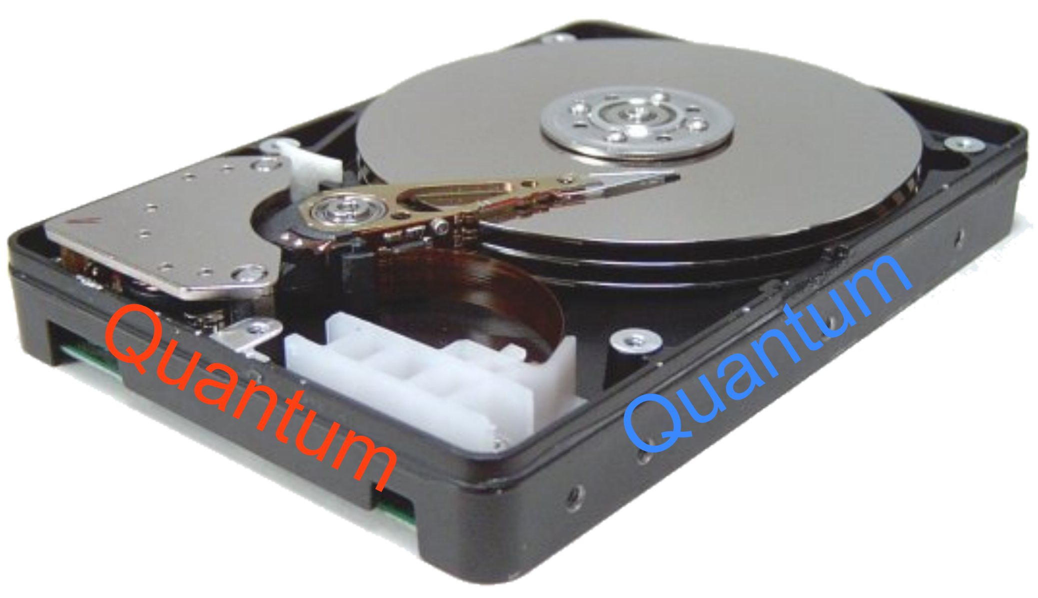


Self-correcting quantum memory

- Can we have self-correcting memory in 3dim?



- Does topological order exist at T>0 ?

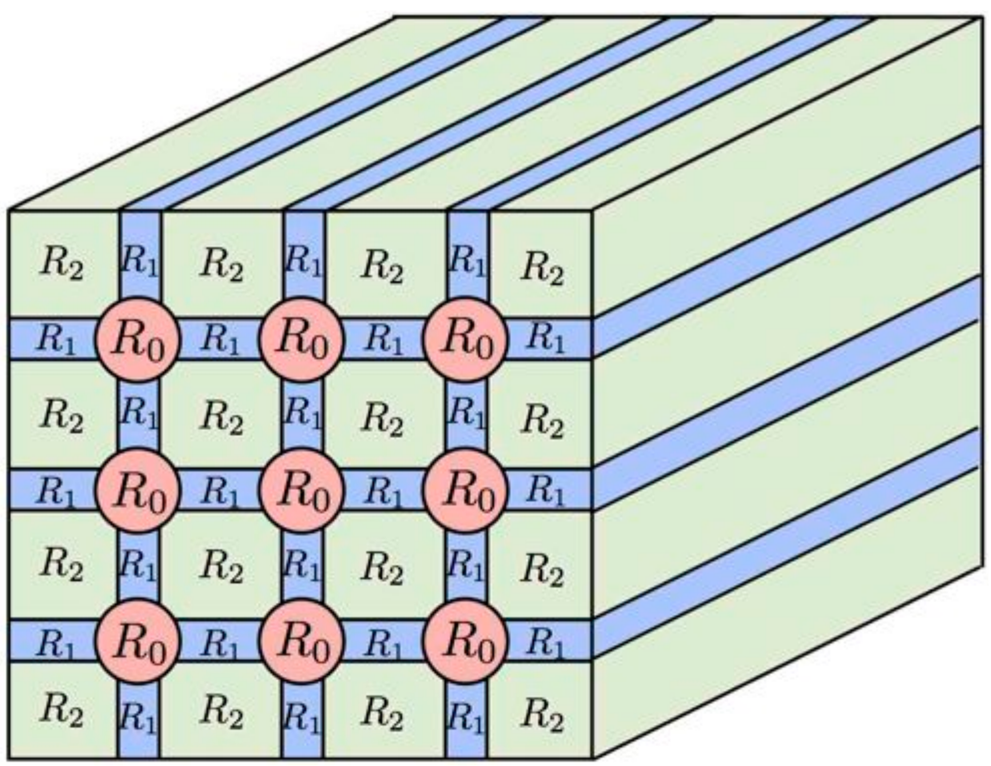


Theorem [Self-correction]

If a stabilizer Hamiltonian in 3 dimensions has fault-tolerantly implementable non-Clifford gates, then the energy barrier is constant.

Proof sketch

- Consider a partition into R_0, R_1, R_2.



- Suppose that there is no string-like logical operators.
- Then, R_0, R_1, R_2 are cleanable, so the code has P_2 (Clifford gate) at most.
- String-like logical operators imply deconfined particles.

Theorem [Self-correction]

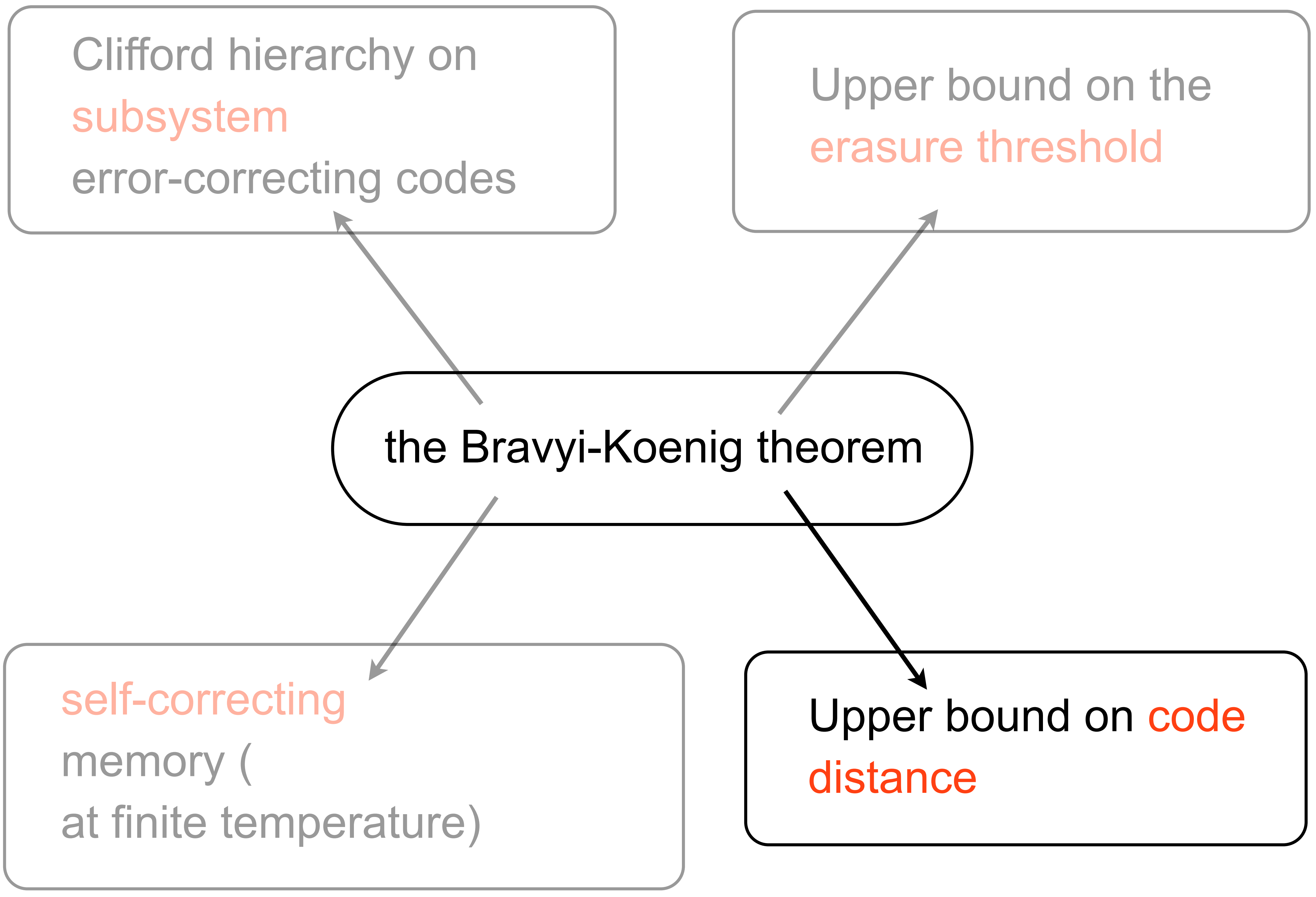
If a stabilizer Hamiltonian in 3 dimensions has fault-tolerantly implementable non-Clifford gates, then the energy barrier is constant.

Remark

- Haah's 3dim cubic code (log(L) barrier) does not have non-Clifford gates.
- Michnicki's 3dim welded code (poly(L) barrier) does not have non-Clifford gates.
- 6-dim color code ((4,2)-construction) has non-Clifford gate and O(L) barrier.

→ * A talk by Brell

Plan of the talk



Theorem [Code distance]

If a topological stabilizer code in D dimensions has a m-th level logical gate, then its code distance is upper bounded by

$$d \leq O(L^{D+1-m})$$

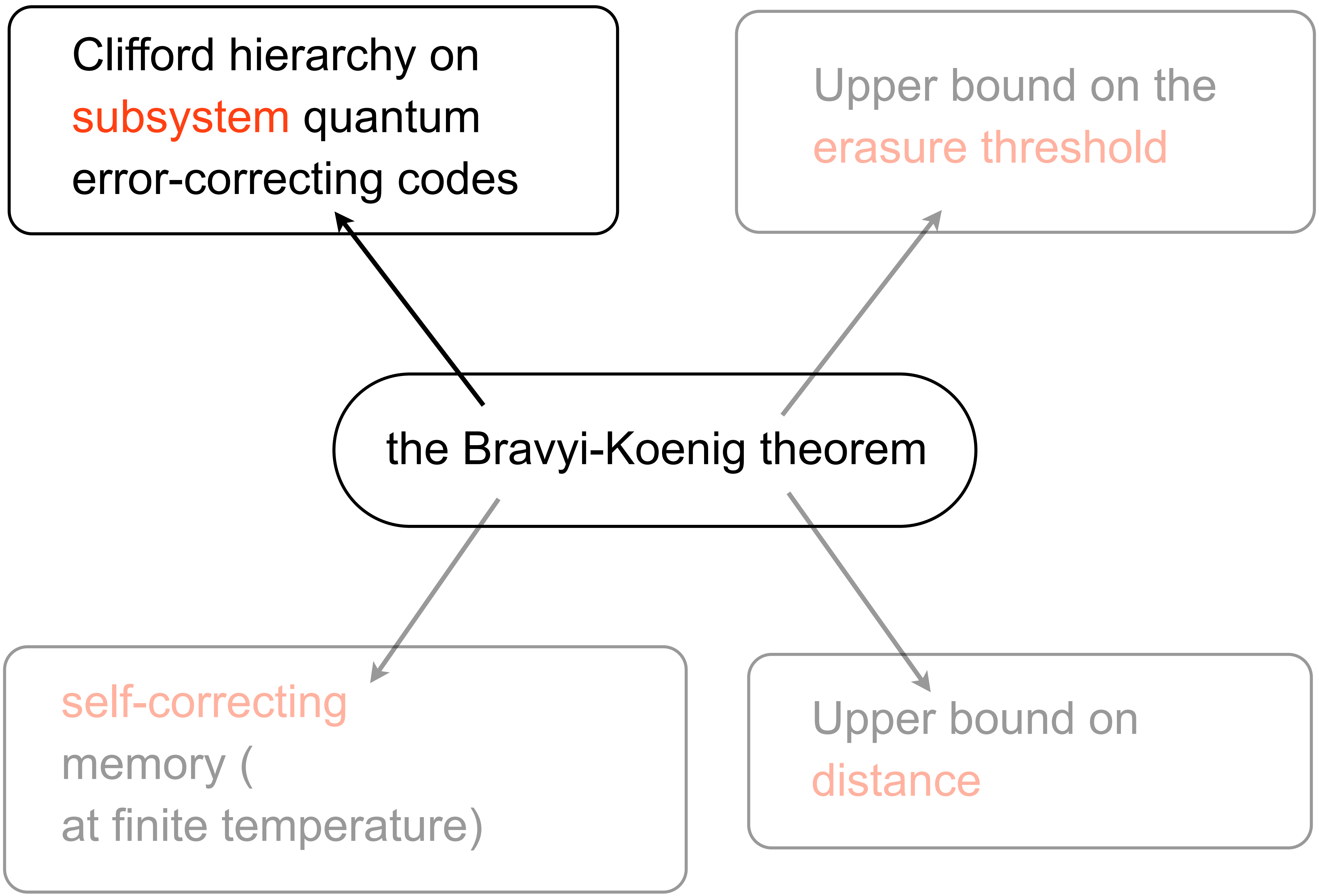
Remark

- Bravyi-Terhal bound for D-dim stabilizer codes (previous best)

$$d \leq O(L^{D-1})$$

- Non-Clifford gate (m>2), our bound is tighter.
- D-dim color code has d=L, saturating the bound.

Plan of the talk



Subsystem code (generalization)

- Starting from non-abelian Pauli subgroup

stabilizer code $\mathcal{S} = \langle S_1, S_2, \ldots \rangle$ $H_{stab} = -\sum_j S_j$

subsystem code $\mathcal{G} = \langle G_1, G_2, \ldots \rangle$ $H_{sub} = -\sum_j G_j$

eg) Kitaev's honeycomb model, Bacon-Shor code, gauge color code

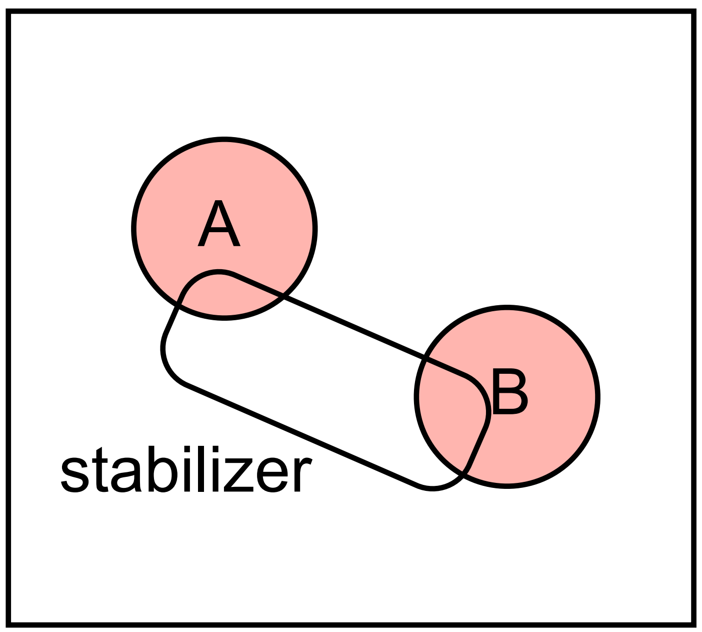
- Subsystem codes require fewer-body terms.

Main result

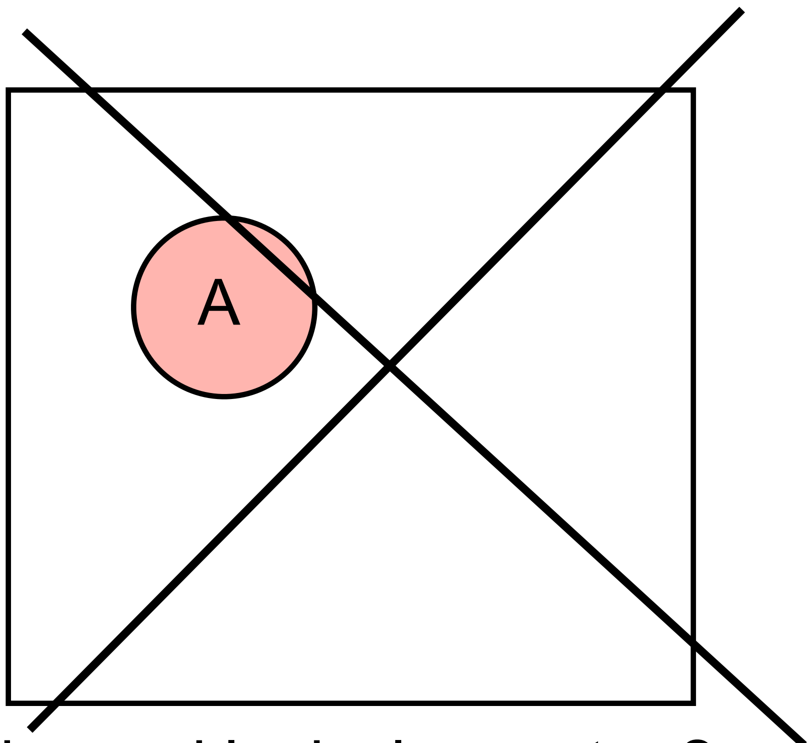
For a D-dimensional subsystem code with local generators, fault-tolerantly implementable logical gates are restricted to $\mathcal{P}_D$ if the code is fault-tolerant.

Breakdown of the union lemma

- The union lemma breaks down.



dressed logical operator



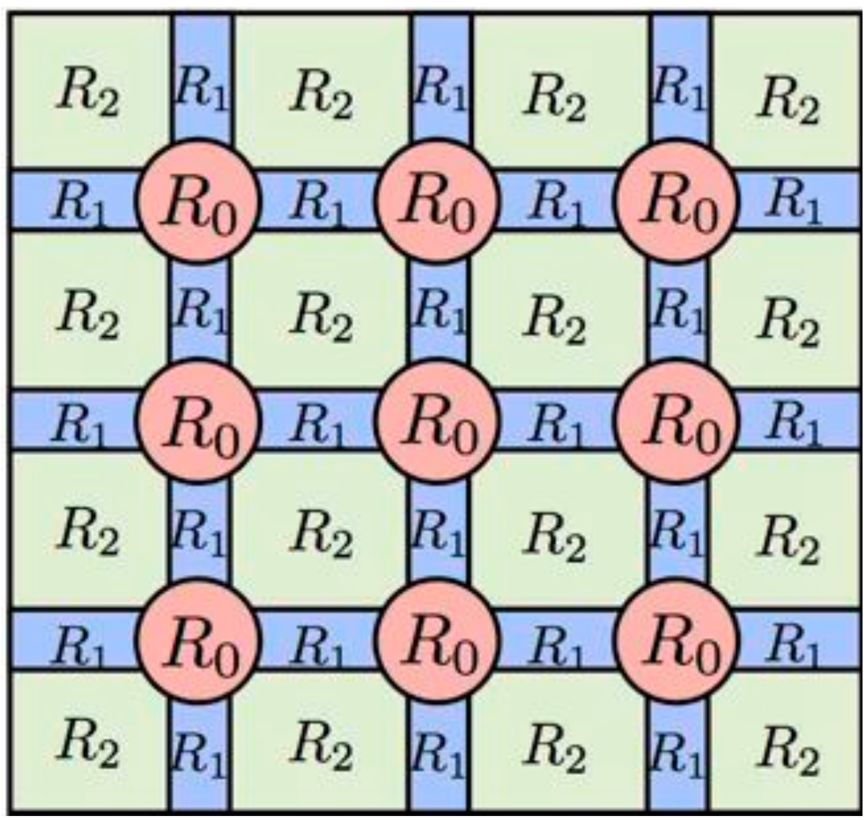
dressed logical operator ?

- Non-local stabilizer operator is closely related to “gapless” spectrum in the Hamiltonian.

Proof of Bravyi-Koenig theorem

- We can split D-dimensional system into D+1 correctable regions.

(eg) 2 dim

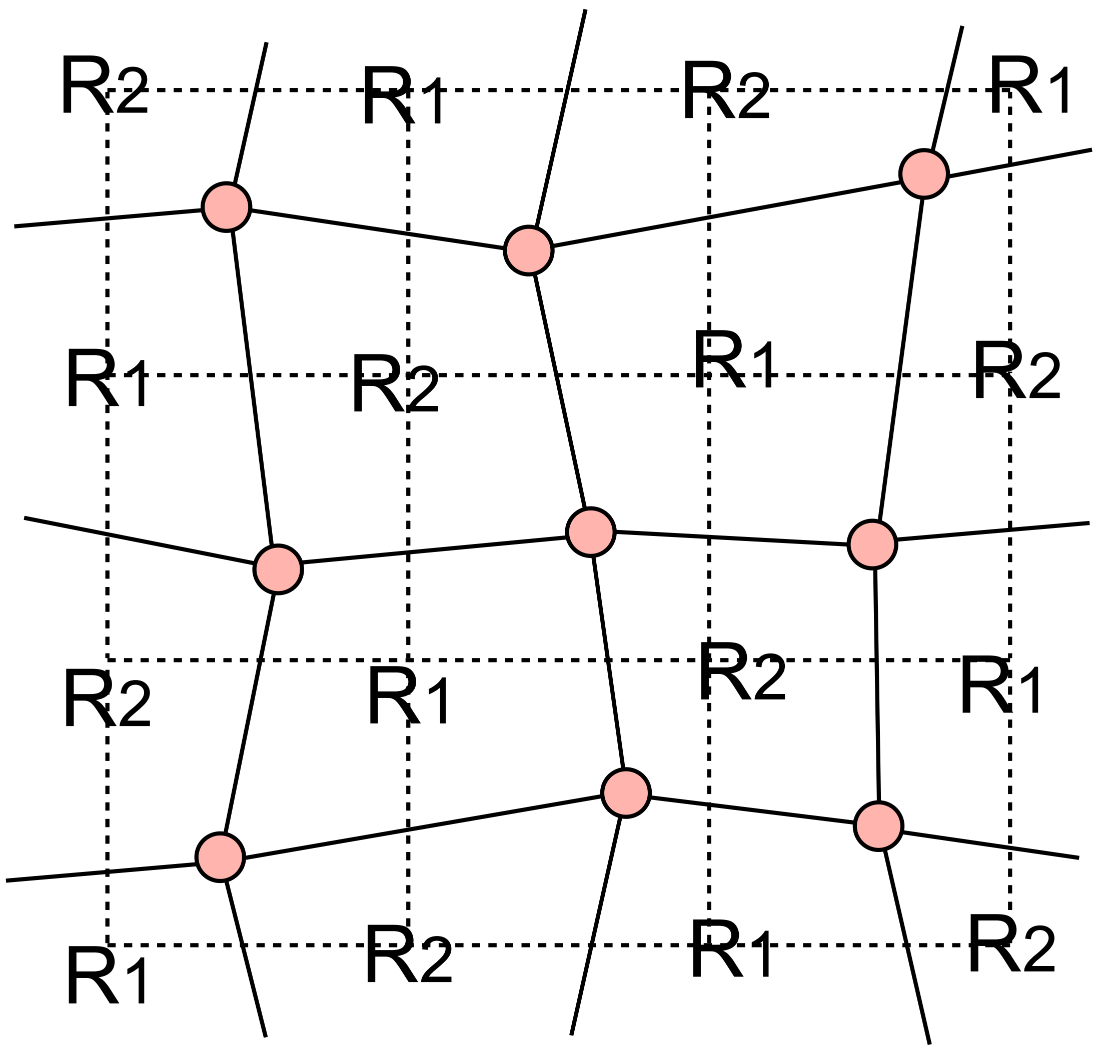


R_0 may not be correctable !

(Each cycle is correctable, but union may not be correctable).

Fault-tolerance of the code

- The code must have a finite error threshold (loss error).



The union of red dots is correctable. (This circumvents the breakdown of the union lemma).

Fault-tolerant logical gates are restricted to P_D.

In D-dimensions, fault-tolerant gates are in P_D.

Summary of the talk

Clifford hierarchy on subsystem quantum error-correcting codes

Upper bound on the erasure threshold

the Bravyi-Koenig theorem

self-correcting quantum memory (topological order at finite temperature)

Upper bound on code distance

Advertisement of our new paper

IQIM

(In)equivalence of the color code and the toric code

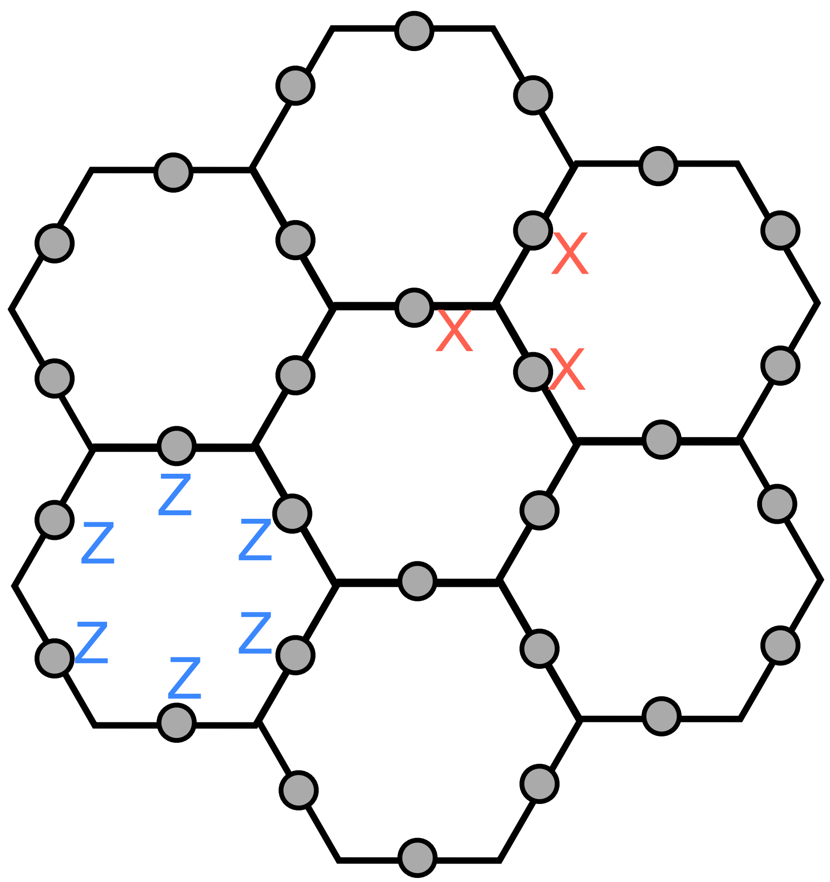
A joint work with

Aleksander Kubica

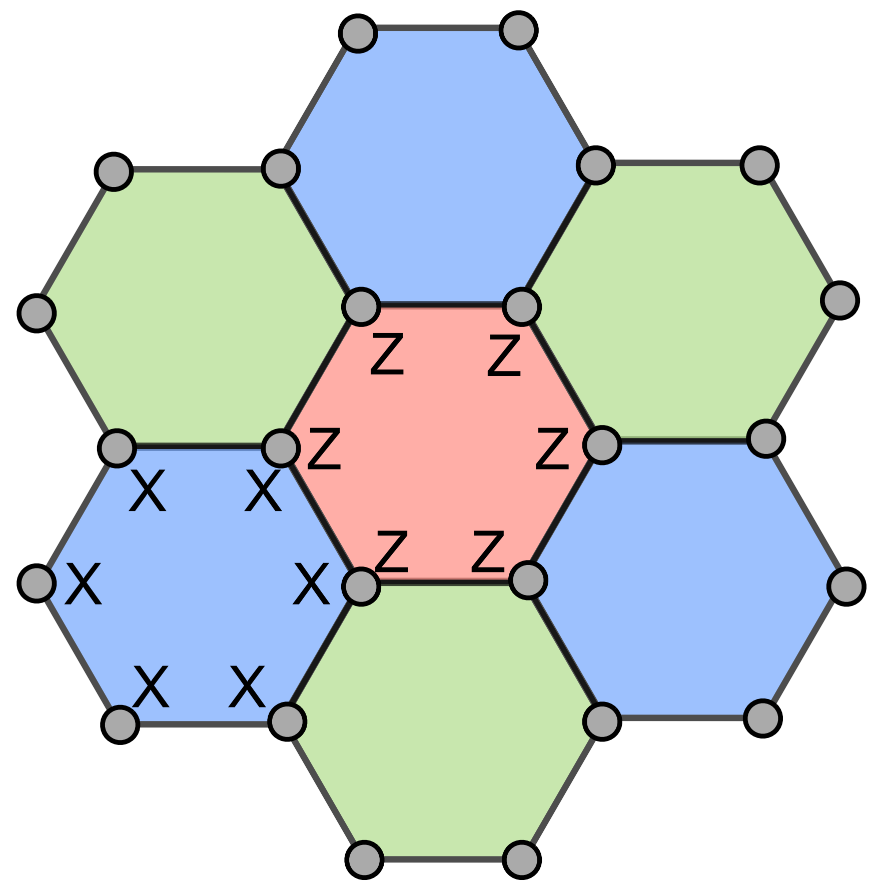
Fernando Pastawski

Toric code vs color code ?

- Similarities and differences between the toric code and the color code ?



toric code



color code

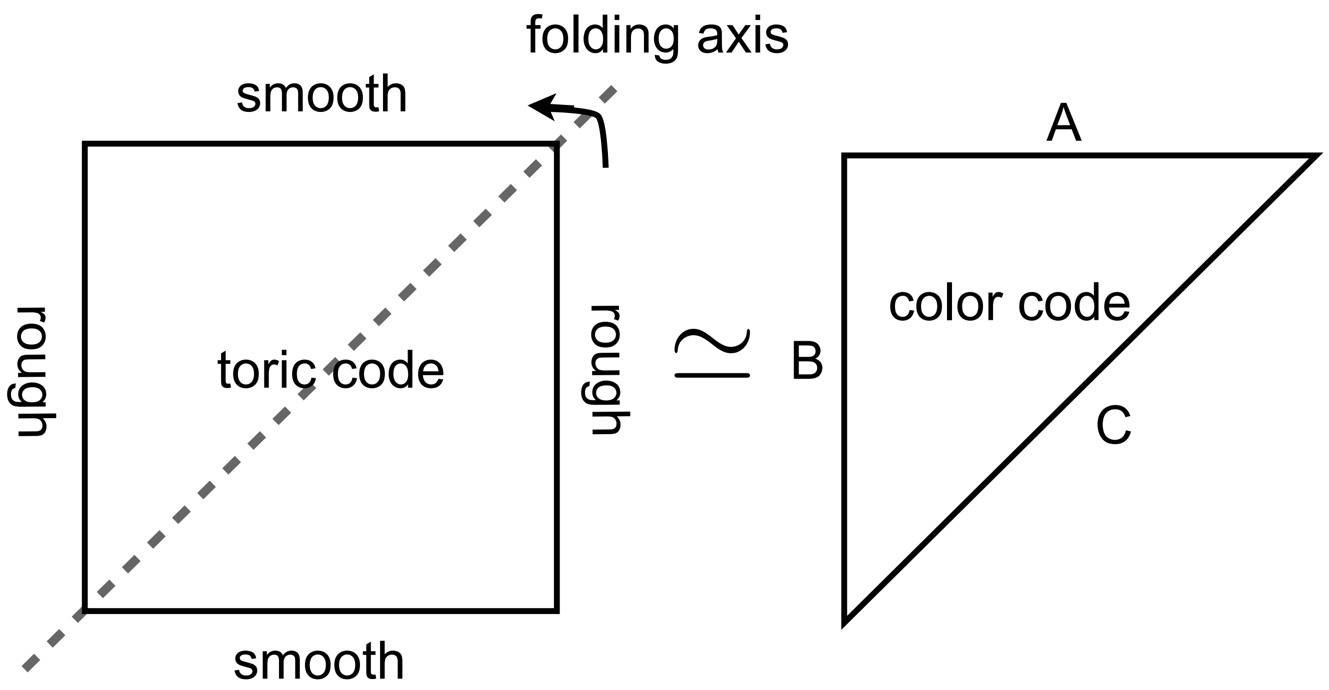
→ * A talk by Bombin

Main results

(1) The d-dim color code on a closed manifold is equivalent to multiple decoupled copies of the d-dim toric code up to a local unitary transformation.

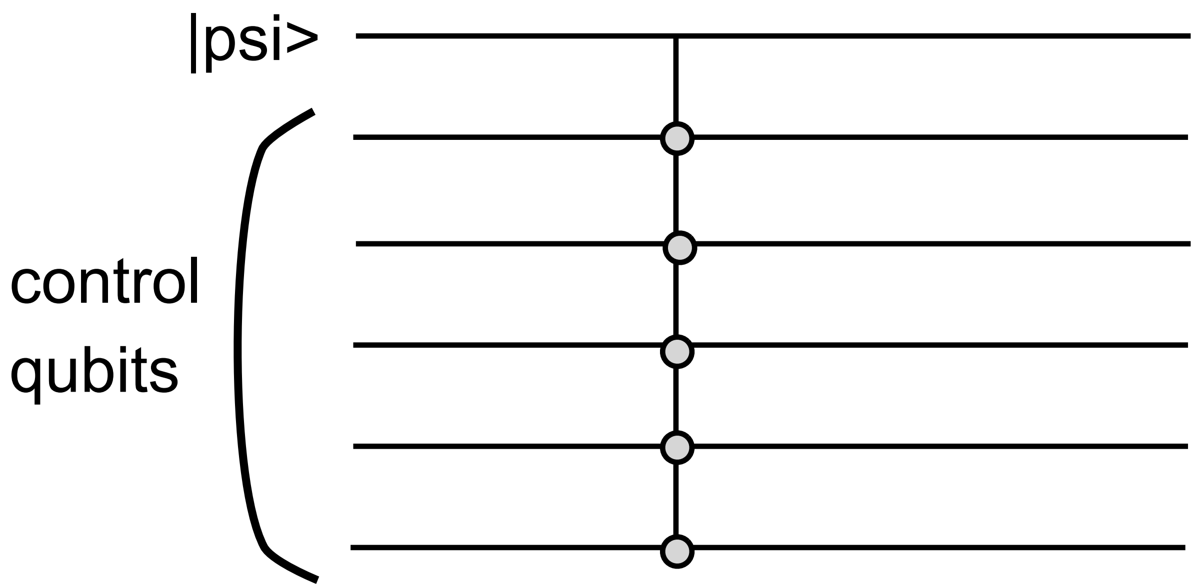
- Extends the known result for 2dim (Yoshida2011)

(2) The 2-dim color code with boundaries is equivalent to the "folded toric code".



Main results (continued...)

(3) Transversal application of Rd gates on the d-dim color code is equivalent to the generalized d-qubit control-Z gate on d decoupled copies of the d-dim toric code.



belongs to Pd

- The toric code saturates the Bravyi-Koenig bound.

Open questions

- Fault-tolerant logical gates in TQFT ? (eg Beverland *et al* 2014)
- The number of transversal gates ? (eg Bravyi & Haah 2012)

  reducing the overhead of magic state distillations
- Non-local, but finite depth unitary ?

  lattice rotations, lattice translations, ...

Many open questions, applications ... ,

Thank you very much.